

## ON EXISTENCE OF DOUBLE COSET VARIETIES

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ABSTRACT. Let  $G$  be a complex affine algebraic group and  $H, F \subset G$  be closed subgroups. The homogeneous space  $G/H$  can be equipped with structure of a smooth quasiprojective variety. The situation is different for double coset varieties  $F \backslash G // H$ . In this paper we give examples showing that the variety  $F \backslash G // H$  does not necessarily exist. We also address the question of existence of  $F \backslash G // H$  in the category of constructible spaces and show that under sufficiently general assumptions  $F \backslash G // H$  does exist as a constructible space.

## 1. INTRODUCTION

Let  $G$  be a complex affine algebraic group and  $H \subseteq G$  be a closed subgroup. By Chevalley Theorem the set of left  $H$ -cosets can be equipped with a uniquely defined structure of a smooth quasiprojective variety such that  $G$  act morphically on  $G/H$ . Moreover, the projection  $G \rightarrow G/H$  is a geometric quotient for the action of  $H$  on  $G$  by right multiplication.

The construction of the homogeneous space  $G/H$  has a natural generalisation: one can take another subgroup  $F \subset G$  and consider double cosets, i. e. the sets  $FgH$ ,  $g \in G$ . These cosets are orbits of the action of  $F \times H$  on  $G$  given by the formula  $(f, h) \circ g = fgh^{-1}$ . It is clear that such action, unlike the action of  $H$  on  $G$  by multiplication, can have orbits of different dimensions, thus it does not necessarily admit a geometric quotient. Because of this we consider a weaker quotient, namely, a categorical one.

The double coset variety  $F \backslash G // H$  is defined to be the underlying space of the categorical quotient  $G \rightarrow F \backslash G // H$  with respect to the described action of  $F \times H$ , if this quotient exists. If the subgroups  $F$  and  $H$  are reductive then this variety exists and coincides with the spectrum  $\text{Spec}({}^F\mathbb{C}[G]^H)$  of the algebra of regular functions on  $G$  invariant under the action of  $F \times H$ . Moreover, if  $G$  is also reductive then by a result of Luna [9] the action  $F \times H : G$  is stable<sup>1</sup>, hence  $F \backslash G // H$  parametrises generic (closed) double cosets.

In this paper we consider the case when the subgroups  $F$  and  $H$  are not reductive. In this setting one can not guarantee that  $F \backslash G // H = \text{Spec}({}^F\mathbb{C}[G]^H)$ ; moreover,  $F \backslash G // H$  does not necessarily exist. To illustrate this we give the following examples:

- I. A unipotent group  $G$  and a subgroup  $U$  of  $G$  such that the variety  $U \backslash G // U$  does not exist.
- II. A reductive group  $G$  and two subgroups  $F, H$  such that the variety  $F \backslash G // H$  does not exist.

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<sup>1</sup>Reductivity of  $G$  is essential: consider the group  $B$  of the upper-triangular matrices and its subgroup  $T$  of the diagonal matrices; the action  $T \times T : B$  is not stable.

III. A semisimple group  $G$  and two subgroups  $F, H$  such that the algebra of  $F \times H$ -invariant regular functions  $R = {}^F\mathbb{C}[G]^H$  is finitely generated and the natural morphism  $\pi : G \rightarrow \text{Spec } R$  is surjective, but  $\pi$  is not a categorical quotient.

It is interesting to remark that though  $U \backslash\backslash G // U$  considered in Example I does not exist as an algebraic variety, it does exist as a constructible space. Thus, here we observe the same phenomenon as in [2], [1] and [5], namely, an action that admits no quotient in the category of algebraic varieties does admit one in the category of constructible spaces.

In Example III the categorical quotient  $F \backslash\backslash G // H$  exists in the category of algebraic varieties; its underlying space is the blow-up of  $\text{Spec}({}^F\mathbb{C}[G]^H)$  at one point; moreover, in this example the categorical quotient separates generic double cosets.

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## 2. PRELIMINARIES ON CATEGORICAL QUOTIENTS

Let an algebraic group  $G$  act on an algebraic variety  $X$ . Recall that the categorical quotient of this action is a  $G$ -invariant (i. e., constant on  $G$ -orbits) morphism  $\pi_G : X \rightarrow Y$  such that every  $G$ -invariant morphism  $\varphi : X \rightarrow Z$  factors uniquely through  $\pi_G$ , that is, there is a unique morphism  $\tilde{\varphi}$  making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Z \\ & \searrow \pi_G \quad \nearrow \tilde{\varphi} & \\ & Y & \end{array}$$

The universal property of  $\pi_G$  implies that  $Y$  is defined uniquely up to isomorphism. Remark that  $\pi_G$  is necessarily surjective. By abuse of language we will sometimes call the variety  $Y = X // G$  the categorical quotient.

If  $G$  is reductive and  $X$  is affine then the categorical quotient for the action  $G : X$  is  $\pi_G : X \rightarrow Y = \text{Spec } \mathbb{C}[X]^G$  with morphism  $\pi_G$  corresponding to inclusion  $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ ; in this case  $\pi_G$  has an important additional property: it separates closed orbits. If  $G$  is not reductive then the quotient  $X // G$  does not necessarily exist. Examples of actions not admitting a categorical quotient are given in [11, 4.3], [1], [2]. Let us point out one example that we will make use of.

**Example 2.1.** [11, 4.3] There is no categorical quotient for the action of a one-dimensional unipotent group  $U$  on space  $\text{Mat}_{2 \times 2}$  of  $2 \times 2$ -matrices given by the formula

$$\lambda \circ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Remark that we have  $\mathbb{C}[\text{Mat}_{2 \times 2}]^U = \mathbb{C}[a_{21}, a_{22}, \det]$  and the canonical morphism  $\pi : \text{Mat}_{2 \times 2} \rightarrow \text{Spec } \mathbb{C}[\text{Mat}_{2 \times 2}]^U \cong \mathbb{A}^3$  separates  $U$ -orbits of generic points having  $a_{21} \neq 0$  or  $a_{22} \neq 0$ . The image of this morphism is  $\mathbb{A}^3$  without the punctured line  $\{a_{21} = a_{22} = 0, \det \neq 0\}$ . Since the image of  $\pi$  is not open, by [2, Corollary 1.4] the action  $U : \text{Mat}_{2 \times 2}$  has no categorical quotient in the category of algebraic varieties.

The morphism  $\pi$  considered in Example 2.1 can be regarded as a quotient morphism after an appropriate modification to the definition of categorical quotient. It turns out that admitting only morphisms into *varieties* as categorical quotients is overly restrictive for

certain actions  $G : X$ . To work around this Bialynicki-Birula introduced in [4] the category of dense constructible subsets. This approach has been further developed in [2] to permit maps into constructible spaces as candidates for quotient morphisms. Recall that a constructible space is a topological space with a sheaf of functions admitting a finite cover by subsets that are isomorphic (as spaces with functions) to constructible subsets of affine varieties. A morphism of constructible spaces is a morphism of spaces with functions. We say that a *constructible quotient* is a categorical quotient in the category of constructible spaces. It is possible for an action  $G : X$  to have no quotient in the category of algebraic varieties, but to have a constructible quotient.

**Example 2.2.** Let a unipotent group  $G$  act on a vector space  $V$ . It follows from [2, Corollary 1.2] that the action  $G : V$  admits a constructible quotient, provided that  $\mathbb{C}[V]^G$  is finitely generated. If  $\rho : V \rightarrow \text{Spec } \mathbb{C}[V]^G$  is the morphism corresponding to the inclusion  $\mathbb{C}[V]^G \subset \mathbb{C}[V]$  then the constructible quotient is  $\rho : V \rightarrow \rho(V)$ . In particular, the map  $\pi$  in Example 2.1 is a constructible quotient for the action  $U : \text{Mat}_{2 \times 2}$ .

Let us point out a fact concerning quotients under two commuting actions; it will be used to identify double coset varieties with quotients of homogeneous spaces. Let  $F \times H$  act on a variety  $X$  and let there be a categorical quotient  $\pi_F : X \rightarrow Y = X//F$ . The group  $H$  acts on  $Y$  as an abstract group: if  $y = \pi_F(x)$  then  $h \circ y = \pi_F(h \circ x)$ . By [3, Theorem. 7.1.4] this action is regular. Moreover, existence of  $Y//H$  is equivalent to existence of  $X//(F \times H)$  and these two quotients coincide:

$$\begin{array}{ccc} X & \xrightarrow{\pi_{F \times H}} & X//(F \times H) = Y//H \\ & \searrow \pi_F \quad \nearrow \pi_H & \\ & Y = X//F & \end{array}$$

The following statement will be used in Proposition 2.

**Lemma 2.3.** *Let an algebraic group  $G$  act on an algebraic variety  $Y$ . Suppose that there is  $y_0 \in Y$  that belongs to closure of every  $G$ -orbit. Consider the action  $G : X \times Y$ , where  $X$  is a normal variety and  $G$  acts trivially on the first factor. Let  $W \subseteq X \times Y$  be a  $G$ -invariant open subset. Suppose that  $W$  contains  $X_0 \times Y$ , where  $X_0 \subseteq X$  is a dense subset, and  $\text{pr}(W) = X$ , where  $\text{pr}$  is the projection onto the first factor. Then the action  $G : W$  has  $\text{pr} : W \rightarrow X$  as a categorical quotient both in the category of algebraic varieties and in the category of constructible spaces.*

*Proof.* Let us fix a  $G$ -invariant morphism  $\varphi : W \rightarrow Z$  into an algebraic variety  $Z$  (resp., into a constructible space) and show that it factors uniquely through  $\text{pr}$ .

Step 1. We claim that  $\varphi$  extends to a continuous map on  $W \cup X \times \{y_0\}$ . Let us fix a point  $(x', y_0) \notin X \times \{y_0\}$  and an arbitrary sequence  $\{x_n\} \subset X_0$  such that  $x_n \rightarrow x'$ . Now we show that the sequence  $\varphi(x_n, y_0)$  converges. Since  $\text{pr}(W) = X$ , there is a point  $(x', y)$  in  $W$  for some  $y \in Y$ . The points  $(x_n, y_0)$  and  $(x_n, y)$  belong to  $W$ , hence by  $G$ -invariance of  $\varphi$  we have  $\varphi(x_n, y) = \varphi(x_n, y_0)$ , thus  $\lim_{n \rightarrow +\infty} \varphi(x_n, y_0) = \lim_{n \rightarrow +\infty} \varphi(x_n, y) = \varphi(x', y)$ . Since a converging sequence can have only one limit,  $\lim_{n \rightarrow +\infty} \varphi(x_n, y_0)$  does not depend on choice of  $(x', y) \in W$ . For the extended map  $\varphi$  we have  $\varphi(x, y) = \varphi(x, y_0)$ , so continuity of  $\varphi|_{X \times \{y_0\}}$  implies continuity of  $\varphi$  on  $W \cup X \times \{y_0\}$ .

Step 2. Now we show that  $X \times \{y_0\}$  can be covered by open affine sets  $X_i \times \{y_0\}$  such that the image of  $\varphi : X_i \times \{y_0\} \rightarrow Z$  is contained in some affine subset of  $Z$ . Let  $\{Z_i\}$  be an affine covering of  $Z$  and  $\{U_i\}$  be an affine covering of  $X$ . The set  $V_{ij} = \varphi^{-1}(\varphi(U_i) \cap Z_j)$  is open in  $U_i$ . Every set  $V_{ij}$  is a union of principal open subsets  $V_{ij} = \cup_k V_{ijk}$ . The sets  $V_{ijk}$  make up the required covering of  $X \times \{y_0\}$ .

Step 3. Since  $\varphi(x, y) = \varphi(x, y_0)$ , we have  $\varphi = \tilde{\varphi} \circ \rho$ , where  $\tilde{\varphi} = \varphi|_{X \times \{y_0\}}$  and  $\rho$  is the map  $W \rightarrow X \times \{y_0\}$ ,  $\rho(x, y) = (x, y_0)$ . Denote  $\iota$  the identification of  $X$  and  $X \times \{y_0\}$ :  $\iota(x) = (x, y_0)$ . We have  $\varphi = \tilde{\varphi} \circ \iota \circ \pi$ , so  $\varphi$  factors through  $\pi$ . It remains to verify that  $\varphi|_{X \times \{y_0\}}$  is a morphism. The variety  $X$  is normal, hence the affine opens  $X_i \times \{y_0\}$  constructed at step 2 are normal varieties, too. Restrictions of  $\varphi$  to these opens are morphisms of affine varieties; if  $Z$  is a constructible space then  $\varphi|_{X_i \times \{y_0\}}$  is a morphism into a constructible set, but it can be regarded as a morphism into an affine variety containing  $\varphi(X_i \times \{y_0\})$ . By theorem on removable singularities the continuous extensions of  $\varphi|_{X_i \times \{y_0\}}$  are morphisms.  $\square$

### 3. EXISTENCE AND NON-EXISTENCE OF DOUBLE COSET VARIETIES

**3.1.** Consider a unipotent group  $G$  and a subgroup  $U$ :

$$G = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & * & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix},$$

here  $*$  denotes an arbitrary number. We claim that if we take  $F = H = U$  then the double coset variety  $F \backslash G // H$  does not exist. Remark that the group  $F \times H = U \times U$  is unipotent, hence every double  $(U, U)$ -coset is closed [11, 1.3]; had  $F$  and  $H$  been reductive, this would have implied existence of the *geometric* quotient  $G \rightarrow G/(F \times H)$ .

**Proposition 1.** *The action  $U \times U : G$  has no categorical quotient in the category of algebraic varieties. It admits a constructible quotient and the constructible quotient parametrises generic double cosets.*

*Proof.* Consider the action of  $G$  on space of  $4 \times 2$ -matrices by left multiplication. The subgroup  $U$  is the stabiliser of the matrix

$$(3.1) \quad M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the homogeneous space  $G/U$  is isomorphic to  $\mathbb{A}^4$  and can be identified with the variety of matrices

$$\begin{pmatrix} * & * \\ * & * \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After this identification the action of  $U$  on  $G/U$  becomes the matrix multiplication; it is therefore isomorphic to the action of  $U$  on space  $\text{Mat}_{2 \times 2}$  of  $2 \times 2$ -matrices by left multiplication. Example 2.1 shows that this action does not admit a categorical quotient. Therefore,  $U \backslash G // U = (G/U) // U$  does not exist.

From Example 2.2 it follows that the action  $U \times U : G$  has a constructible quotient  $\pi : G \rightarrow U \backslash\backslash G // U \subset \mathbb{A}^3$  which separates generic double cosets.  $\square$

**Remark.** The constructible quotient  $\pi : G \rightarrow U \backslash\backslash G // U$  does not separate all closed double cosets. Indeed, all  $2 \times 2$ -matrices with  $a_{21} = a_{22} = 0$  (we use notation of Example 2.1) are fixed under the action of  $U$  and have  $\det = 0$ , hence their preimages in  $G$  are closed  $(U, U)$ -cosets, which are mapped by  $\pi$  to  $0 \in \mathbb{A}^3$ .

**3.2.** Take  $G = GL_4$  and consider the action of  $G$  on  $4 \times 2$ -matrices. Let  $H$  be the stabiliser of the matrix  $M$ ,  $M$  being the same as in 3.1. The homogeneous space  $W = G/H$  is identified with the variety of  $4 \times 2$ -matrices with non-zero columns. Let  $F$  be the subgroup of  $G$  consisting of the following matrices:

$$\begin{pmatrix} 1 & a & 0 & 0 \\ & 1 & 0 & 0 \\ & & s & 0 \\ & & & s \end{pmatrix}, \quad a \in \mathbb{C}, \quad s \in \mathbb{C}^\times.$$

The subgroup  $F$  acts on  $W$  via matrix multiplication.

**Proposition 2.** *The action  $F \times H : G$  does not admit a categorical quotient in the category of algebraic varieties, but has a constructible quotient.*

*Proof.* The group  $F$  is a direct product  $F = U \times S$  of one-dimensional unipotent group  $U$  and one-dimensional torus  $S$ . The categorical quotient for the action  $S : W$  is  $\text{pr} : W \rightarrow \text{Mat}_{2 \times 2}$ , where the projection  $\text{pr}$  erases the bottom half of matrices of  $W$ . Indeed, one can apply Lemma 2.3 with the acting group  $S$  and  $X = Y = \text{Mat}_{2 \times 2}$  representing top and bottom halves of matrices respectively, and with  $X_0$  consisting of matrices with non-zero columns.

Thus, had the quotient  $W // F = F \backslash\backslash GL_4 // H$  existed, it would have been also  $(W // S) // U = \text{Mat}_{2 \times 2} // U$ , but, according to Example 2.1, the latter quotient does not exist.

By Lemma 2.3 and Example 2.2, the actions  $S : W$  and  $U : \text{Mat}_{2 \times 2}$  both have a constructible quotient, thus  $F \backslash\backslash GL_4 // H = (W // S) // U$  exists as a constructible space.  $\square$

**3.3.** This example is based on [2, 4.5]. Consider the following symmetric bilinear form on  $\mathbb{C}^4$ :  $(e_1, e_4) = (e_2, e_3) = 1$  and the other pairings of basis vectors are zero. The cone

$$X = \{\mathbf{x} \in \mathbb{C}^4 \mid x_1x_4 + x_2x_3 = 0\} \setminus \{(0, 0, 0, 0)\}$$

is the collection of non-zero isotropic vectors, therefore  $X = SO_4 / H$ , where  $H$  is the stabiliser of a non-zero isotropic vector. As  $F$  we take the following unipotent subgroup of  $SO_4$ :

$$\begin{pmatrix} 1 & a & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -a \\ & & & 1 \end{pmatrix}, \quad a \in \mathbb{C}.$$

The algebra  $\mathbb{C}[X]^F$  is freely generated by  $x_2$  and  $x_4$ ; indeed, these two functions are  $F$ -invariant and generic orbits meet the plane  $\{x_1 = x_3 = 0\}$ , so there are no other generators. It is clear that the canonical morphism  $\pi : X \rightarrow \text{Spec } \mathbb{C}[X]^F = \mathbb{A}^2$  is surjective. Nevertheless,  $\text{Spec } \mathbb{C}[X]^F$  is not the quotient for  $F : X$  because the following morphism  $\varphi : X \rightarrow \mathbb{P}^1$  does not factor through  $\pi$ :

$$\varphi(x_1, x_2, x_3, x_4) = (x_2 : x_4) = (x_1 : -x_3).$$

Indeed,  $\pi(x_1, x_2, x_3, x_4) = (x_2, x_4)$ , and from  $\varphi = \tilde{\varphi} \circ \pi$  it follows that  $\tilde{\varphi}(x_2, x_4) = (x_2 : x_4)$  when  $x_2 \neq 0$  or  $x_4 \neq 0$ , hence  $\tilde{\varphi}$  is not continuous in  $(0, 0)$ , which is not possible.

Let us show that the considered action has a categorical quotient, though it does not coincide with  $\text{Spec } \mathbb{C}[X]^F$ .

**Proposition 3.** *The action  $F : X$  has a categorical quotient in the category of algebraic varieties.*

*Proof.* The quotient is the blow-up of the origin in  $\mathbb{A}^2$

$$\hat{\mathbb{A}}^2 = \{((x, y), (u : v)) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xv - yu = 0\}$$

with the morphism  $\rho : X \rightarrow \hat{\mathbb{A}}^2$ ,  $\rho(x_1, x_2, x_3, x_4) = ((x_2, x_4), (x_1 : -x_3)) = ((x_2, x_4), (x_2 : x_4))$ .

Let us check that every  $F$ -invariant morphism  $\varphi : X \rightarrow Z$  factors through  $\rho$ . By  $F$ -invariance of  $\varphi$  we have  $\varphi(cx_1, x_2, cx_3, x_4) = \varphi(x_1, x_2, x_3, x_4)$ . Indeed, if one of  $x_2, x_4$  is not zero then the point  $(x_1, x_2, x_3, x_4)$  and the points  $(cx_1, x_2, cx_3, x_4)$  belong to one orbit of  $F$ ; if  $x_2 = x_4 = 0$  then we have

$$\begin{aligned} \varphi(x_1, 0, x_3, 0) &= \lim_{t \rightarrow 0} \varphi(x_1, tx_1, x_3, -tx_3) = \\ &= \lim_{t \rightarrow 0} \varphi(cx_1, tx_1, cx_3, -tx_3) = \varphi(cx_1, 0, cx_3, 0). \end{aligned}$$

Define  $\tilde{\varphi} : \hat{\mathbb{A}}^2 \rightarrow Z$  as the morphism taking  $((x, y), (u : v))$  to  $\varphi(u, x, -v, y)$ . The reasoning above shows that  $\tilde{\varphi}$  is well defined. Thus,  $\varphi = \tilde{\varphi} \circ \rho$ , i. e.,  $\varphi$  factors through  $\rho$ . Since  $\rho(X) = \hat{\mathbb{A}}^2$ , the morphism  $\tilde{\varphi}$  can be chosen uniquely.  $\square$

**Remark.** It is clear that  $\rho : X \rightarrow \hat{\mathbb{A}}^2$  separates orbits of points having  $x_2 \neq 0$  or  $x_4 \neq 0$ . However,  $\rho$  does not separate all closed orbits: the points  $z = (x_1, 0, x_3, 0)$  and  $z' = (cx_1, 0, cx_3, 0)$  are  $F$ -fixed, but  $\rho(z) = \rho(z')$ . Thus, the quotient  $q : \text{SO}_4 \rightarrow F \backslash \text{SO}_4 // H = \hat{\mathbb{A}}^2$  separates generic double cosets, but fails to separate all closed double cosets.

**3.4.** Remark that in Examples I and II the actions  $F \times H : G$  have no categorical quotient in the category of algebraic varieties but do admit one in the category of constructible spaces.

**Question.** Let  $G$  be a connected affine algebraic group and  $F, H$  be closed subgroups in  $G$ . Is it true that  $F \backslash G // H$  exists as a constructible space?

The following proposition gives a partial answer to this question.

**Proposition 4.** *Let  $G$  be a connected affine algebraic group and  $F, H \subset G$  be closed connected subgroups with trivial character groups. Suppose that the algebra  ${}^F\mathbb{C}[G]^H$  is finitely generated and let  $\pi : G \rightarrow \text{Spec}({}^F\mathbb{C}[G]^H)$  be the canonical morphism. Then  $F \backslash G // H$  exists as a constructible space and the map  $\pi : G \rightarrow \pi(G)$  is the constructible quotient for the action of  $F \times H$  on  $G$ .*

*Proof.* By [10, Theorem 6], the underlying variety of  $G$  has a finite divisor class group. Additionally,  $F$  and  $H$  have trivial character groups, therefore every  $F \times H$ -invariant hypersurface  $D \subset G$  is the zero set of an invariant function  $f_D \in {}^F\mathbb{C}[G]^H$ . It follows from [2, Corollary 1.2] that the action  $F \times H : G$  has  $\pi : G \rightarrow \pi(G)$  as a constructible quotient.  $\square$

**Remark.** One can often give a positive answer to the question on finite generation of  ${}^F\mathbb{C}[G]^H$ . Recall that if  $R$  is a reductive group,  $Z$  is an affine  $R$ -variety and  $U \subset R$  is a maximal

unipotent subgroup then the algebra  $\mathbb{C}[Z]^U$  is finitely generated [8, Chapter 3.2]. Thus, the constructible space  $F \backslash G // H$  is guaranteed to exist if both groups  $F$  and  $H$  are maximal unipotent subgroups in bigger reductive subgroups  $F', H' \subseteq G$  or if one of them is semisimple and the other one is a maximal unipotent subgroup in a bigger reductive subgroup. Other results on finite generation of algebras of invariants can be found in [6].

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